13.3 Measurement Tools for Curves

Today we will cover arc length and curvature focusing on how to compute the formulas and giving some motivation.

Preview of tools

Arc Length

$$s(t) = \int_0^t |\boldsymbol{r}'(u)|du = \text{distance (arc length)}$$

Curvature

3D Formula: $\kappa(t) = \dfrac{|\boldsymbol{r}'(t) \times \boldsymbol{r}''(t)|}{|\boldsymbol{r}'(t)|^3}$

2D Formula: $\kappa(x) = \dfrac{|f''(x)|}{\left(1+\left(f'(x)\right)^2\right)^{3/2}}$

TNB-Frame (we will discuss in 13.4)

Part 1: Arc Length (i.e. Distance) HW 13.3 / 1–3

Sp'19 – Exam 1 – Loveless

Find the distance along the curve

$$\boldsymbol{r}(t) = \langle t^3, 3t^2, 6t \rangle$$

from (0, 0, 0) to (1, 3, 6).

Visual: https://desmos.com/3d/h75znwwviz

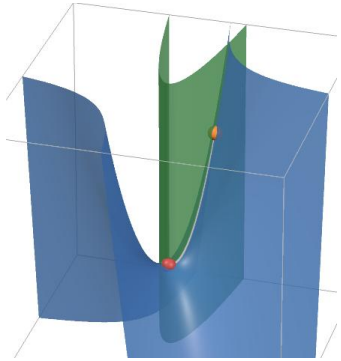
Example 1 HW 13.3 / 7

My slinky is a helix that has a radius of 1.4 inches, it has 39 turns, and it is 2.5 inches tall when collapsed. How long is it?



Example 2 HW 13.3 / 3

Let C be the curve of intersection of the parabolic cylinder $x^2 = 2y$ and the surface $3z = xy$. Find the length of this curve from the origin to the point $(5, 25/2, 125/6)$.



Arc length parameterization HW 13.3 / 4–5

Distance traveled along the curve:

$$s(t) = \int_0^t |\boldsymbol{r}'(u)| du$$

Solve for t in terms of s, then substitute into **r**(t).

Example: $\boldsymbol{r}(t) = \langle 1 + 3t, 2t, 5t \rangle$

Find s(t), measured from t = 0 in the direction of increasing t.

Reparameterize the curve in terms of arc length.

Distance along a curve HW 13.3 / 4–5

Using the same curve, find the point 5 units along the curve from (1, 0, 0), in the direction of increasing t.

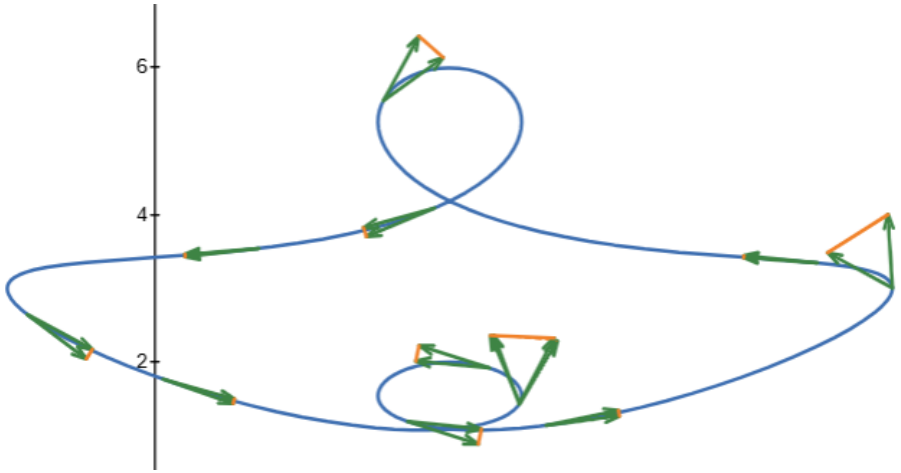
Curvature

The curvature at a point, K, is a measure of how quickly a curve is changing direction at that point.

$$\frac{change\ in\ direction}{change\ in\ distance} = \left|\frac{d\vec{\boldsymbol{T}}}{ds}\right|$$

Roughly, how much does your direction change if you move a small amount ("one inch") along the curve?

$$\left|\frac{\overrightarrow{\boldsymbol{T_2}} - \overrightarrow{\boldsymbol{T_1}}}{"one\ inch"}\right| = \left|\frac{\Delta\vec{\boldsymbol{T}}}{\Delta s}\right|$$



Visual: https://desmos.com/3d/ilgaxncxz6

This is NOT easy to compute directly from arc length, so instead we use this the following shortcut

$$K(t) = \left|\frac{d\vec{\boldsymbol{T}}}{ds}\right| = \frac{|\vec{\boldsymbol{r}}'(t) \times \vec{\boldsymbol{r}}''(t)|}{|\vec{\boldsymbol{r}}'(t)|^3}$$

(see the last page of these notes for proof or look in the text)

Sp'19 – Exam 1 – Loveless Like HW 13.3 / 6

Find the curvature of

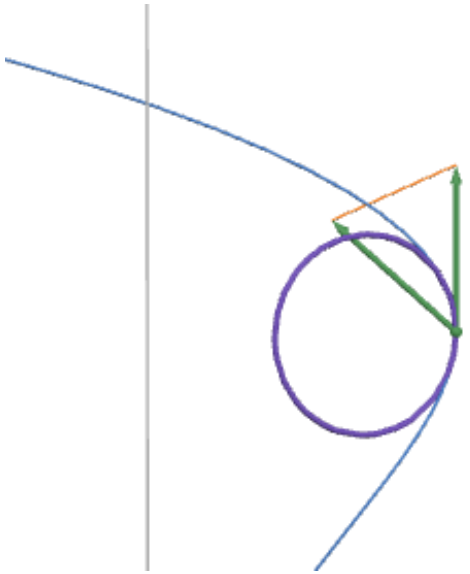
$$\boldsymbol{r}(t) = \langle t^3, 3t^2, 6t \rangle$$

at t = 0.

Cool Side Fact

The radius of curvature is the radius of the circle that would best fit this curve.

$$\text{radius of curvature} = \frac{1}{\kappa}.$$



Visual: desmos.com/3d/rbjm1twka2

In 2D the curvature formula becomes

$$K(x) = \frac{|\boldsymbol{r}' \times \boldsymbol{r}''|}{|\boldsymbol{r}'|^{\mathbf{3}}} = \frac{|f''(x)|}{\left(1 + \left(f'(x)\right)^2\right)^{3/2}}$$

(for proof, see the last page of these notes)

Example

Find the x-value at which curvature is maximum for $f(x) = x^2$.

Does your answer make sense?

What if you change to a different function?

Proofs of curvature formulas:

Proof of 3D shortcut formula (feel free to ignore)

Write v = $|\mathbf{r}'(t)|$. Since $\mathbf{r}'$ = v$\mathbf{T}$, the product rule gives

$$\boldsymbol{r}'' = v'\boldsymbol{T} + v\boldsymbol{T}'.$$

Take the cross product with $\mathbf{r}'$ = v$\mathbf{T}$. Note that $\boldsymbol{T} \times \boldsymbol{T}$ = $\mathbf{0}$, since they are the same direction, so we get…

$$\boldsymbol{r}' \times \boldsymbol{r}'' = v^2(\boldsymbol{T} \times \boldsymbol{T}').$$

Since $|\mathbf{T}|$ = 1 and $\mathbf{T} \cdot \mathbf{T}'$ = 0 (so the angle is 90 degrees),

$$|\boldsymbol{T} \times \boldsymbol{T}'| = |\boldsymbol{T}||\boldsymbol{T}'|\sin(90°) = |\boldsymbol{T}'|$$

Thus,

$$|\boldsymbol{r}' \times \boldsymbol{r}''| = v^2|\boldsymbol{T}'|, \text{so } |\boldsymbol{T}'| = \frac{|\boldsymbol{r}' \times \boldsymbol{r}''|}{v^2}$$

By definition

$$\kappa = \frac{|\boldsymbol{T}'|}{v}$$

And replacing $|\boldsymbol{T}'| = \frac{|\boldsymbol{r}' \times \boldsymbol{r}''|}{v^2}$ gives the shortcut formula:

$$\kappa = \frac{|\boldsymbol{r}' \times \boldsymbol{r}''|}{v^3} = \frac{|\boldsymbol{r}' \times \boldsymbol{r}''|}{|\boldsymbol{r}'|^3}$$

Proof for 2D formula (again feel free to ignore)

For y = f(x), form the vector function

$$\boldsymbol{r}(x) = \langle x, f(x), 0\rangle$$
$$\boldsymbol{r}'(x) = \langle 1, f'(x), 0\rangle$$
$$\boldsymbol{r}''(x) = \langle 0, f''(x), 0\rangle$$

Taking the cross-product gives…

$$\boldsymbol{r}' \times \boldsymbol{r}'' = \langle 0,0, f''(x)\rangle$$
$$|\boldsymbol{r}' \times \boldsymbol{r}''| = |f''(x)|$$
$$|\boldsymbol{r}'(x)| = \sqrt{1 + f'(x)^2}$$

Thus,

$$K(x) = \frac{|\boldsymbol{r}' \times \boldsymbol{r}''|}{|\boldsymbol{r}'|^{\boldsymbol{3}}} = \frac{|f''(x)|}{\left(1 + \left(f'(x)\right)^2\right)^{3/2}}$$

Proof sketch for the Arc Length formula

Step 1: Approximate the curve by short line segments.

Split a ≤ t ≤ b into small intervals.

On one interval, the length can be approximated by

$$L_i = \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2 + (\Delta z_i)^2}.$$

Over a short time interval Δt, we can write…

$$\sqrt{(\Delta x_i)^2 + (\Delta y_i)^2 + (\Delta z_i)^2} \cdot \frac{\Delta t}{\Delta t} =$$
$$\sqrt{\left(\frac{\Delta x_i}{\Delta t}\right)^2 + \left(\frac{\Delta y_i}{\Delta t}\right)^2 + \left(\frac{\Delta z_i}{\Delta t}\right)^2} \cdot \Delta t$$

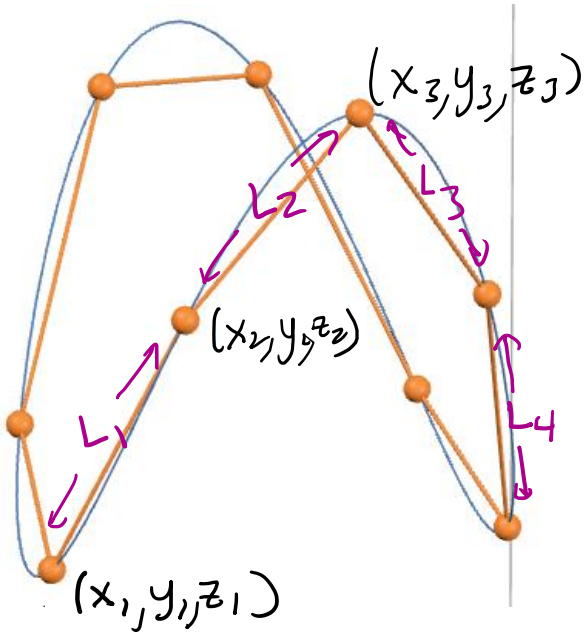
Step 2: Add up the pieces, then take a limit.

$$L \approx \sum_{i=1}^{n} \sqrt{\left(\frac{\Delta x_i}{\Delta t}\right)^2 + \left(\frac{\Delta y_i}{\Delta t}\right)^2 + \left(\frac{\Delta z_i}{\Delta t}\right)^2} \cdot \Delta t \ .$$

As the interval width approaches zero, $\frac{\Delta x_i}{\Delta t}$ approaches $x'(t_i)$, $\frac{\Delta y_i}{\Delta t}$ approaches $y'(t_i)$ and $\frac{\Delta z_i}{\Delta t}$ approaches $z'(t_i)$. Thus, the sum on the right approaches an integral:

$$L = \int_a^b \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\,dt = \int_a^b |\boldsymbol{r}'(t)|\,dt$$

Note: This derivation assumes **r** has a continuous derivative on [a, b].



In 2D

$$L = \int_a^b \sqrt{x'(t)^2 + y'(t)^2}\,dt$$

For a graph y = f(x), use x as the parameter:

$$\boldsymbol{r}(x) = \langle x, f(x)\rangle,$$
$$\boldsymbol{r}'(x) = \langle 1, f'(x)\rangle.$$

Thus, the familiar formula from Math 125

$$L = \int_a^b \sqrt{1 + f'(x)^2}\,dx$$